Characterization of soliton damping in the granular chain under gravity

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A soliton created in the horizontal granular chain damps due to gravity in the vertical chain. We show that there are two types of propagating modes, quasisolitary and oscillatory, in the vertical chain, depending on the strength of impulse. We find that the type of damping is a power law in depth or time. We also find that the absolute value of the exponent of the power law decreases as the strength of the initial impulse increases in the quasisolitary regime. In the oscillatory regime, however, in which the initial impulse is weak, the power-law exponent is independent of the strength of the initial impulse. We show that the power-law damping is caused by the gravitation which results in the change of the force constant at each contact.

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The dynamics of granular materials are very useful in many applications [1,2] and important as a new emerging area of physics. In addition to interesting properties, such as hydrodynamic flow, pattern formation, and clustering [3], the study of the propagation of sound or elastic impulse in the granular medium is also useful and interesting [4,5] in connection with obtaining information inside the granular medium. The granular chain with the nonlinear Hertzian contact [6] has been studied by Nesterenko [7], who showed that the propagating mode of the initial impulse in a highly nonlinear regime is a soliton. The solitonlike behavior of the signal in two-dimensional granular beds has been studied numerically by Sinkovits and Sen [8,9]. The properties of a solitary signal in the chain of iron balls have been examined experimentally by Coste *et al.* [10] recently.

Recently, MacKay [11] proved the existence of solitary waves in the horizontal Hertzian chain using a rather general mathematical theorem given by Friesecke and Wattis [12]. This theorem is applied to the equation of motion $\ddot{q}_n = V'(\phi_n) - V'(\phi_{n-1})$ where $\phi_n = q_{n+1} - q_n$, but is restricted to the one-dimensional lattice. The proof of MacKay may be compared with that of Nesterenko, who used the Taylor expansion to get the soliton equation of motion for the highly nonlinear regime. Even though the geometrical effect [13] may be important in a real situation, the one-dimensional granular system rather easily shows the fundamental physics existing in the nonlinear granular chain.

Recently, Sen *et al.* [14] performed an interesting study on the propagation and the backscattering of the elastic signal in the Hertzian chain under gravity. This work suggested a possible way to detect a buried impurity using a solitonlike wave. A fundamental study has been done by Hong *et al.* [15], who have shown analytically the existence of the power-law behaviors of the signal propagating down the gravitationally compacted granular chain with arbitrary power-law type contact force. That work was restricted to the so-called oscillating regime which is achieved by weak impulses. But some interesting things still remain in examining the extended regime of impulse in the nonlinear granular chain under gravity.

The gravity effect is essential in dispersing the soliton in the gravitationally compacted chain. An interesting discovery we make is that a solitary wave created in the horizontal chain damps under the effect of gravity and the type of damping is a power-law type. We find that the depth-dependent power-law behavior of the propagating signal is generic for the whole range of strength of the impulse. The power-law exponent depends on the strength of the impulse when the power-law type of the contact force law of the grain has been determined. The impulse-dependent behavior of the power-law exponent shows interesting features. For rather strong impulses, the absolute value of the power-law exponent decreases, i.e., the signal becomes more solitary, as the initial impulse increases. This phenomenon is understood as follows. The role of gravity becomes negligible as the impulse becomes stronger. The state under strong impulse is similar to that of the horizontal chain in which a soliton is the propagating mode [7,11]. In fact, the propagating mode of a strong impulse under gravity is quite similar to that of a soliton in the horizontal chain except for damping. Therefore, we call this mode the quasisoliton or the quasisolitary wave.

An interesting phenomenon we discover in this work is that the absolute value of the power-law exponent increases and approaches a saturated value as the strength of the initial impulse decreases. The propagating mode in this regime is oscillatory and we call this the oscillatory regime. The power-law exponents are the same for different strengths of impulse in the oscillatory regime. A previous work [15] showed impulse-independent power laws for the limit of small oscillation. The power-law behaviors in both quasisolitary and oscillatory regimes have never been studied in other works as far as we know.

The special feature of the system under consideration has two aspects. One is the nonlinear contact force and the other is the effect of gravity. The former is common to both horizontal and vertical granular chains, while the latter is a characteristic of the vertical chain. We study the propagating properties in the quasisolitary regime where the soliton suffers damping due to gravity. We analyze how the gravity affects soliton propagation when the gravity is introduced in the vertical granular chain. Even though the previous work [15] has shown the power-law behaviors in the small oscillation limit, it is not clear why the power-law behaviors are common in any strength of impulse in the vertical chain.

This work may supply an understanding of the signal propagation in the vertical granular chain with a nonlinear contact force.

In this work, we focus on the motion of grains in a vertical granular chain with a nonlinear contact force of arbitrary power-law type. It is usually hard to treat nonlinear problems analytically, so we use molecular dynamics simulations to study the effect of gravity in the grain motion. We solve numerically the equation of motion of a grain at z_i which is the distance from the top of the chain to the center of the ith spherical grain, such as

$$m\ddot{z}_n = \eta[\{\Delta_0-(z_n-z_{n-1})\}^p-\{\Delta_0-(z_{n+1}-z_n)\}^p]+mg, \tag{1}$$

where m is the mass of the grain, Δ_0 is the distance between adjacent centers of the spherical grain, p is the exponent of the power-law type contact force, and η is the elastic constant of the grain under consideration. Therefore, the overlap between the adjacent grains at nth contact is $\delta_n=\Delta_0-(z_{n+1}-z_n)$. We do not consider the plastic deformation in treating Eq. (1). For the Hertzian chain, i.e., $p=3/2$, the equation of motion comes from the Hertzian interaction energy between neighboring granular spheres which is given by [6]

$$V(\delta_n)=\frac{2}{5D}\left(\frac{R_nR_{n+1}}{R_n+R_{n+1}}\right)^{1/2}\delta_n^{5/2}\equiv b\,\delta_n^{5/2}, \tag{2}$$

where R_n is the radius of the spherical grain and

$$D=\frac{3}{4}\left(\frac{1-\sigma_n^2}{E_n}+\frac{1-\sigma_{n+1}^2}{E_{n+1}}\right), \tag{3}$$

where σ_n, σ_{n+1} and E_n, E_{n+1} are Poisson's ratios and Young's moduli of the bodies at neighboring positions, respectively [6]. Therefore, $\eta=(5/2)b$ for the Hertzian chain.

To perform numerical simulations for Eq. (1), we choose a vertical chain of $N=2\times10^3$ grains. We choose 10^{-5} m, 2.36×10^{-5} kg, and 1.0102×10^{-3} s as the units of distance, mass, and time, respectively. These units gives the gravitational acceleration $g=1$ [14]. We set the grain diameter 100, mass 1, and the constant b of Eq. (1) 5657 for the molecular dynamics simulation. The equilibrium condition

$$mgn=\eta\delta_n^p \tag{4}$$

has been used for the $(n+1)$th grain of the vertical chain. Using the third-order Gear predictor-corrector algorithm [16] as a calculational tool, we perform numerical simulations for $p=3/2$ as an example. Even though the criterion for the initial impulse neglecting the plastic deformation [10,17] and viscoelastic dissipation to make Eqs. (1) and (2) valid as a model, we do not apply any restriction to the initial impulse, because it only plays the role of parameter for the solitariness of the signal in this work. Therefore, we choose various impulses for our study and show the change of solitariness of the signal as the strength of impulse changes.

Figure 1 shows the snapshots of two typical types of the grain velocity signals propagating down the vertical chain

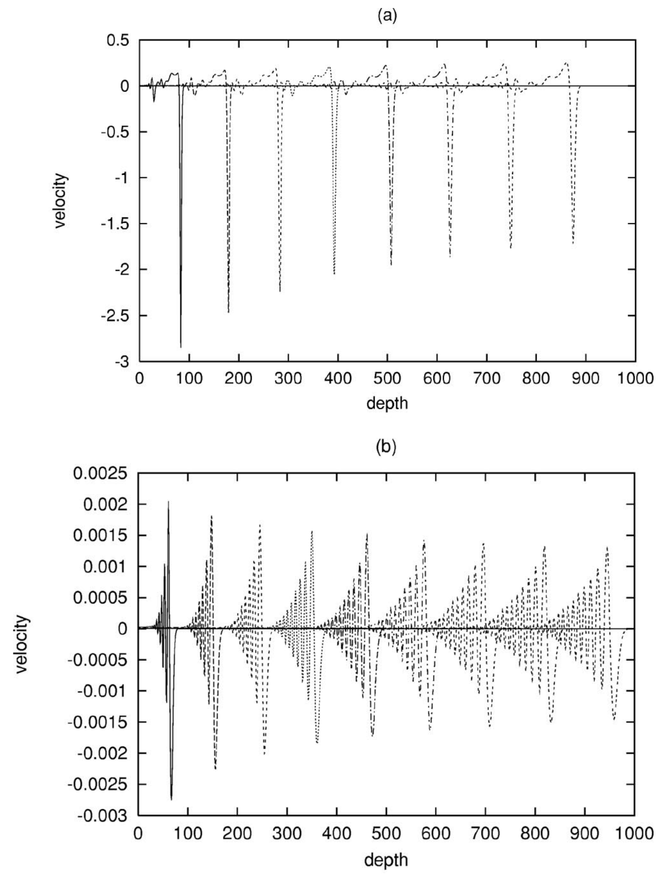

FIG. 1. Snapshots of typical modes of propagating signals under strong and weak impulses in a gravitationally compacted granular chain with Hertzian contact law. The program units of velocity have been used. The unit of depth is the number of grains. (a) Quasisolitary mode due to strong impulse $v_i=6$. (b) Oscillatory mode due to weak impulse $v_i=0.01$.

with Hertzian contact. Figure 1(a) obtained for the initial impulse velocity $v_i=6$ belongs to the quasisolitary regime, while Fig. 1(b) obtained for $v_i=0.01$ belongs to the oscillatory regime. The quasisolitary signal shown in Fig. 1(a) has the same propagating characteristics, such as increasing propagating speed, dispersion, and decreasing grain velocity, as those of oscillating signal in Fig. 1(b). We plan to study these properties of the quasisolitary signal in future; this will require a great deal of work. We treat only grain velocities in both regimes in this work.

Figure 2(a) shows the depth-dependent behavior of the leading grain velocity peaks for various initial impulses expressed by velocity. Both quasisolitary and oscillatory signals damp in power law with depth. One can see that the lower three, $v_i=0.1$, 0.01, and 0.001, corresponding to a weak impulse, have the same depth-dependent behavior. This can be more clearly seen in Fig. 2(b), which plots the power-law exponents (negative) versus inverse v_i. The logarithmic scale has been used for the abscissa of the graph.

A remarkable feature is seen in Fig. 2(b). That is, one can separate the propagating behavior into two classes, the varying exponent regime of $v_i>1$ and the flat regime of $v_i<1$ where the exponent is $-1/4$ [15] for the Hertzian chain. The former corresponds to the quasisolitary regime where the velocity signal shown in Fig. 1(a) is a typical propagating

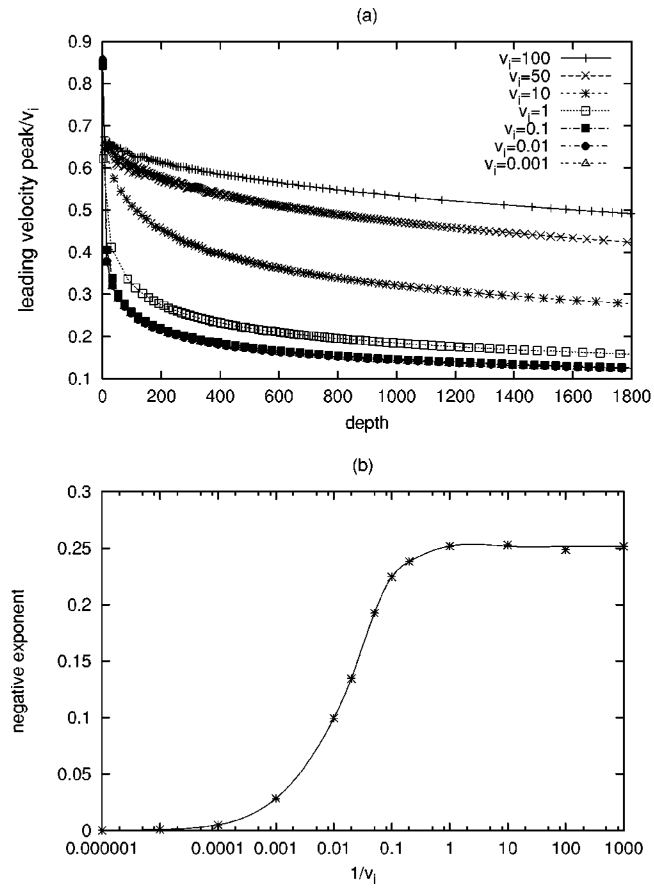

FIG. 2. (a) Depth-dependent behaviors of the leading velocity peaks for various v_i. Data for $v_i=0.1$, 0.01, and 0.001 are overlapped. (b) The behavior of power-law exponents of the leading velocity peak for various v_i's. The ordinate is the absolute value of the exponent and the abscissa is the inverse of v_i in logarithmic scale. Crosses are data points and the solid line is a guide for the eye. The same units as Fig. 1 have been used.

mode. The latter, on the other hand, corresponds to the oscillatory regime in which the velocity signal shown in Fig. 1(b) is a typical propagating mode. Figure 2 shows that the damping of the signal becomes weaker and weaker as the initial impulse strengthens. This implies that the pulse type velocity signal created in the quasisolitary regime becomes a soliton in the limit of large impulse.

It is interesting to see the reason the power-law exponent is v_i independent in the oscillatory regime, and v_i dependent in the quasisolitary regime. Since v_i determines the displacement of the signal, one may understand the v_i dependence and v_i independence behavior by studying the role of displacement in the equation of grain motion. For this purpose, we introduce a new variable ψ_n, denoting the displacement of the nth grain from equilibrium, defined by

$$\psi_n = z_n - n\Delta_0 + \sum_{l=1}^{n} \left(\frac{mgl}{\eta}\right)^{1/p}, \tag{5}$$

where the last term is the sum of overlaps up to nth contact and we set $z_0=\psi_0=0$. Equation (1) can be transformed into

$$m\ddot{\psi}_n = \eta\left[\left(\frac{mgn}{\eta}\right)^{1/p} + (\psi_{n-1}-\psi_n)\right]^p - \eta\left[\left\{\frac{mg(n+1)}{\eta}\right\}^{1/p} + (\psi_n-\psi_{n+1})\right]^p + mg \tag{6}$$

using Eq. (5).

For the oscillatory regime, the condition

$$|\psi_{n-1}-\psi_n| \ll \left(\frac{mgn}{\eta}\right)^{1/p} \tag{7}$$

is valid and the expansion gives Eq. (6) as

$$m\ddot{\psi}_n = -\mu_n(\psi_n-\psi_{n-1}) + \mu_{n+1}(\psi_{n+1}-\psi_n), \tag{8}$$

where $\mu_n = mpg(\eta/mg)^{1/p} n^{1-1/p}$ is the force constant of nth contact. Both the left- and right-hand sides of Eq. (8) are linear in ψ_n. Therefore, the scaling analysis tells us that the equation of motion (8) has nothing to do with the initial impulse v_i.

For the quasisolitary regime, however, the condition

$$(\psi_{n-1}-\psi_n) \gg \left(\frac{mgn}{\eta}\right)^{1/p} \tag{9}$$

is valid and the expansion gives Eq. (6) as

$$\begin{aligned} m\ddot{\psi}_n = {}& \eta[(\psi_{n-1}-\psi_n)^p - (\psi_n-\psi_{n+1})^p] \\ & - \eta p[\delta_{n+1}(\psi_n-\psi_{n+1})^{p-1} \\ & - \delta_n(\psi_{n-1}-\psi_n)^{p-1}] + mg, \end{aligned} \tag{10}$$

where $\delta_n = (mgn/\eta)^{1/p}$ denotes the overlap at nth contact. Different order of ψ_n in the left and right sides of Eq. (10) implies that v_i dependence must appear in the signal characteristics.

We now focus on the quasisolitary regime in which the signal is described by Eq. (10). This nonlinear differential equation may not be solved analytically for an arbitrary p. But one can appreciate the role of each term of Eq. (10). The first term on the right side of Eq. (10) which is the leading term of the expansion is the same as the contact force of the horizontal chain in which a solitary wave is created under strong enough impulse if $p>1$. The existence of the solitary wave in the horizontal granular chain for $p>1$ may be proved by extending the work of MacKay [11] for $p=3/2$ to arbitrary p. This will be reported in a separate work [18]. The second term, on the other hand, is the contact force with renormalized force constant $p\eta(mgn/\eta)^{1/p}$ which is varying at each contact of the horizontal chain with a nonlinear contact force. Since the effect of gravity is already immersed in the second term of Eq. (10) via changing variables of Eq. (5), the constant mg in the last term of Eq. (10) does not play a crucial role under the condition of Eq. (9). Therefore, one can understand that the gravity nearly exhausts its role in changing the force constant at each contact.

We have shown analytically in the previous paper [15] that the variation of the force constant at each contact yields the power-law damping in the signal propagation. This has been performed in the limit of small oscillation regime and the transformed equation of motion for displacement was linear. We understand, therefore, that the power-law behavior does not result from the nonlinearity of the equation of motion but from the variation of force constant at each contact. Even though we may not draw the power-law behavior analytically from the second term of Eq. (10) for arbitrary p because of its nonlinearity, it is clear from the above argument that the second term is the only source of the power-law behavior.

In conclusion, we have seen the types of damping of the signals going down the gravitationally compacted nonlinear chain in which the propagating feature is quite different from that of the horizontal chain where the soliton is the propagating mode. We find that there are two different regimes, quasisolitary and oscillatory, in which the propagating characteristics are different from each other and the power-law behavior in depth is a generic property appearing under gravity. We have also seen that the effect of gravity competes with that of impulse, i.e., a stronger impulse produces a more solitary wave. In the quasisolitary or strong impulse regime, a solitary wave damps due to gravity in the form of a power law in depth and the absolute value of the power-law exponent decreases as the initial impulse increases. In the oscillatory or weak impulse regime, however, we observe that the power-law exponent approaches a saturated value as the initial impulse weakens and the exponent becomes independent of the strength of the impulse eventually. The analytic scheme of the previous work [15] has been applied to this regime. As a final remark, we understand that the power-law damping of a solitary wave in depth is due to gravity, which induces the change in force constant at each contact in the vertical granular chain.

We did not provide a concrete analytical explanation of the role of the initial impulse, even though we provide numerical results, such as Fig. 2. The amplitude of displacement, the length of signal, the peak grain velocity, and the propagating speed of the signal do depend on the strength of initial impulse in the quasisolitary regime. Analyzing these in an analytical way requires more work. We plan to do this in future.

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